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THE KERNEL OF A NONSTANDARD GAME.(U)

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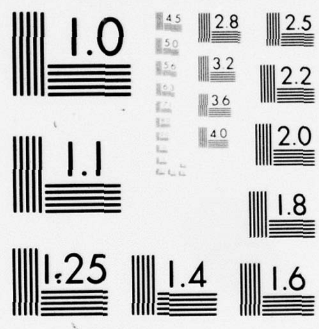

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① LEVEL II

6 THE KERNEL OF A NONSTANDARD GAME

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Introduction

An extension of the classical games of the Von Neumann-Morgenstern variety, involving a finite number of participants and transferable utility, to a nonstandard *Finite context is provided. By choosing the *Finite set to be of the form $[0,\omega]$, for an infinite integer $\omega \in N^* - N$, the construction allows the treatment of denumerably infinite games by essentially finite means by imbedding them externally within the appropriate *Finite set. In general, such games will not have non-empty Kernels, where the use of Kernel in this context is that of the syntactic equivalent of the Kernel for standard finite games. It then follows that solution concepts that are in close relationship to the Kernel, such as the Bargaining Set, and the Nucleolus, are in general not non-empty in their equivalent syntactic forms.

A related concept, that of the Quasi-Kernel, $QK^*(\Gamma^*)$, is shown to be, in general, non-empty in the nonstandard context for the class of *Finite games with Q-bounded payoff. We provide an example in Section III which serves to indicate that $QK^*(\Gamma^*)$ is a strictly weaker solution concept than the syntactic equivalent of the Kernel, $K^*(\Gamma^*)$

A more complete treatment of the Quasi-Kernel in addition to discussions of related solution concepts to the Quai-Kernel can be found in A Nonstandard Theory of Games, Parts I and II, Harvard University, Center on Decision and Conflict in Complex Organizations, 1979.

Notation

R	The standard real numbers
R_+	The nonnegative real numbers (standard)
N	The standard natural numbers
R^*	The nonstandard real numbers
N^*	The extended natural numbers in R^*
F^*	An internal *Finite set in N^* : $F^* \subseteq N^*$ and $\lVert F^* \rVert = \omega$
$\underset{\not\sim}{>}$	Greater than by at least an infinitesimal amount
$\underset{\not\sim}{>>}$	Greater than by a noninfinitesimal amount
$\geq$	Equal to or greater than by at least an infinitesimal amount
$(=)_{\text{Mod } M_1}$	Of at most an infinitesimal difference
ω	An infinite integer
$\langle F^*, A(F^*), V^* \rangle$	A *Finite cooperative game for $F^* = [0,\omega]$; $v^* : (P(F))^* \to R^*$; $A(F^*)$ the algebra of internal subsets of F^*, i.e. sets in $(P(F))^*$. We assume that $v^*(\phi) = 0$ and that v^* is superadditive on $A(F^*)$ with Q-bound
(x^*, F^*)	The set of payoff configurations for the game $\langle F^*, A(F^*), v^* \rangle$. By definition, $(x^*,F^*) = \left\{ x^* \varepsilon (R_+^*)^{F^*} : \left(\sum_{j \varepsilon F^*} x^*(j) = v^*(F^*) \right)_{\text{Mod } M_1} \right\}$

$QK^*(\Gamma^*)$ — The set of payoffs $x^* \varepsilon (x^*,F^*)$ for the game Γ^* such that $\left[\tilde{S}^*_{ij}(x^*) = \tilde{S}^*_{ji}(x^*)\right]_{\text{Mod } M_1}$ a.e. in F^* where a.e. in F^* has the interpretation that the set in F^* for which $\left[\tilde{S}^*_{ij}(x^*) \neq \tilde{S}^*_{ji}(x^*)\right]_{\text{Mod } M_1}$ is negligible in the following sense,

$$\left(\frac{\|S\|}{\|F^*\|} = 0\right)_{\text{Mod } M_1}$$

T^*_{ij} — $\{S \,\varepsilon\, A(F^*) = i \,\varepsilon\, S \ldots j \notin S\}$

$\tilde{e}(S,x^*)$ — The excess of $S \,\varepsilon\, A(F^*)$ with respect to the payoff configuration $x^* \varepsilon (x^*,F^*)$. By definition, $\tilde{e}(S,x^*) = v^*(S) - \sum_{j \varepsilon S} x^*(j)$

$$\tilde{S}^*_{ij}(x^*) = \max_{S \,\varepsilon\, T^*_{ij}} \tilde{e}(S,x^*)$$

M_0 — The set of numbers in R^* that are finite - $M_0 = \left\{x^* \varepsilon R^* : |x^*| < r \text{ for some } r \,\varepsilon\, R_+ - \{0\}\right\}$

M_1 — The set of numbers in R^* that are infinitesimal - $M_1 = \left\{x^* \varepsilon R^* : |x^*| < r \text{ for all } r \,\varepsilon\, R_+ - \{0\}\right\}$

S-topology — The topology on R^* generated by the sets $S(x^*,R) = \left\{y^* : |x^* - y^*| < r \text{ for some } r \,\varepsilon\, R_+ - \{0\}\right\}$

Q-topology — The topology on R^* generated by the sets $Q(x^*,R) = \left\{y^* : |x^* - y^*| < r \text{ for some } r \,\varepsilon\, R^*_+ - \{0\}\right\}$. One observes that the Q-topology is stronger than the S-topology, i.e. $S \subsetneq Q$.

Q-closed (open)	Closed (open) in the Q-topology.
Q-bounded	Bounded by $r \varepsilon R^*_+$
Q-convex hull	For a set $B \subseteq (R^*)^n$, $n \varepsilon N^*$, Q-con(B) is the set of combinations $\sum_{j=1}^{n} \alpha_j x^*_j$, $\sum_{j=1}^{n} \alpha_j = 1$, $\alpha_j \varepsilon R^*_+$, $\alpha_j \varepsilon [0,1]$, $x^*_j \varepsilon B$ for each $j = 1,\ldots,n$. Then B is Q-convex if Q-con(B) $\subseteq$ B.
$u(x^*)$	The monad of a point $x^* \varepsilon R^*$. By definition, $u(x^*) = \left\{y^* \varepsilon R^* : (x^* = y^*)_{\text{Mod } M_1}\right\}$
$st(x^*)$	The standard part of x^*. By definition, $st(x^*) = \left\{r \varepsilon R : (x = r)_{\text{Mod } M_1}\right\}$ and is uniquely defined for $x^* \varepsilon M_0$. $st(x^*)$ is not defined for $x^* \notin M_0$.

I. *Finite Games and the Quasi-Kernel

Let M be a set theoretical structure, sufficiently rich to generate the real number system and allow M^* to be an $\aleph_1$-saturated enlargement of M in the sense of Robinson [4]. Then M^* generates a non-Archimedean extension of R, R^*, which is termed a nonstandard model of analysis. All references to nonstandard objects will be assumed to be with respect to M^*.

By a *Finite cooperative game we will mean a triplet, $\langle F^*, A(F^*), v^*\rangle = \Gamma^*$ where $F^* = [0,\omega] \subseteq N^*$ is an internal *Finite set, $A(F^*)$ is the algebra of internal subsets of F^*, and v^* is a nonnegative Q-bounded set function from $A(F^*)$ to R_+^* such that

(i) $v^*(\phi) = 0$

(ii) $v^*(F^*) \leq K^*$ for some $K^* \in N^*$

(iii) $v^*(S \cup T) \geq v^*(S) + v^*(T)$ if $S \cap T = \phi$

A pre-imputation is an internal assignment, $x^* : F^* \to R^*$ such that $\left[\sum_{j \in F^*} x^*(j) = v^*(F^*)\right]_{\text{Mod } M_1}$ and such that $(\forall j \in F^*)(x^*(j) \in M_0^+)$.

An imputation, or what we shall term a payoff configuration, is a pre-imputation that satisfies, in addition, individual rationality, that is, $(\forall j \in F^*)(x^*(j) \geq 0)$. We implicitly assume that the value of any single person coalition is null in this characterization of payoff configurations, that is, $(\forall j \in F^*)(v^*(\{j\}) = 0)$. The set of payoff configurations will be denoted as (x^*, F^*), assuming,

for the sake of simplicity, that the coalition structure is simply the grand coalition, F^*.

Definition I.1: We will define the notion of one player being stronger than another player in the game Γ^* in terms of the following concepts. Let $i, j \in F^*$, $i \neq j$. Then we define

(i) $\bar{e}(S,x^*) = v^*(S) - \sum_{j \in S} x^*(j)$ for $S \in A(F^*)$ and $x^* \in (x^*,F^*)$

(ii) $T^*_{ij} = \{S \in A(F^*) : i \in S \ldots j \notin S\}$

(iii) $\bar{S}^*_{ij}(x^*) = \max_{S \in T^*_{ij}} \bar{e}(S,x^*)$ for $x^* \in (x^*,F^*)$

Then a player, i, is said to be stronger than a player j, with respect to a given payoff configuration, $x^* \in (x^*,F^*)$, symbolized as $i \textcircled{>} j$ if it is the case that $\bar{S}^*_{ij}(x^*) \gg \bar{S}^*_{ji}(x^*)$ and $x^*(j) \geq 0$. Two players, i and j, are said to be equipollent, symbolized as $i \textcircled{=} j$, if $\sim(i \textcircled{>} j)$ and $\sim(j \textcircled{>} i)$; that is, neither player outweighs the other.

Definition I.2: Let the following relation be defined with respect to Γ^* for $\bar{x}^* \in (x^*,F^*)$: For two players $i, j \in F^*$, let $i\, \bar{R}(\bar{x}^*)\, j$ mean that $i \textcircled{>} j$ with respect to $\bar{x}^* \in (x^*,F^*)$.

Lemma I.3: The relation $\bar{R}(x^*)$ satisfies the following properties:

(1) $\tilde{R}(x^*)$ is acyclic.

(2) $\tilde{R}(x^*)$ is open in (x^*,F^*). That is, the set $\{y^* : i\,\tilde{R}(x^*)\,j\}$ is Q-open in (x^*,F^*) for $i,\ j \in F^*$.

(3) If $x^* \in (x^*,F^*)$ and $x^*(j) = 0$ for some $j \in F^*$, then $(\sim\exists i \in F^*)(i\,\tilde{R}(x^*)\,j)$.

Proof: Property (3) is straightforward.

Property (2) follows easily from the observation that for fixed $S \in A(F^*)$, $\tilde{e}(S,x^*)$ and therefore $\tilde{S}_{ij}(x^*)$ is Q-continuous in x^*. The set $\{x^* \in (x^*,F^*) : [\tilde{S}^*_{ij}(x^*) - \tilde{S}_{ji}(x^*)] \gg_{\ell} 0 \ \wedge\ x^*(j) >_{\ell} 0\}$ is obviously Q-open and for $y^* \in u(x^*)$ taking x^* as fixed, it follows that $[\tilde{S}^*_{ij}(y^*) - \tilde{S}^*_{ji}(y^*)] \gg_{\ell} 0$ and $y^*(j) >_{\ell} 0$, which gives the result.

To show acyclicity, property (1), it is sufficient to show that $\tilde{R}(x^*)$ is transitive. The following proof is due to Maschler (unpublished Lecture Notes).

If $i\,\tilde{R}(x^*)\,j$ and $j\,\tilde{R}(x^*)\,k$, then $i\,\tilde{R}(x^*)\,k$ is to be shown. One has, therefore, by assumption, that $[\tilde{S}^*_{ij}(x^*) \gg_{\ell} \tilde{S}^*_{ji}(x^*)]$ and $[\tilde{S}^*_{jk}(x^*) \gg_{\ell} \tilde{S}^*_{kj}(x^*)]$ while $x^*(j) >_{\ell} 0$ and $x^*(k) >_{\ell} 0$. It remains to show that $[\tilde{S}^*_{ik}(x^*) \gg_{\ell} \tilde{S}^*_{ki}(x^*)]$.

$$\text{Let } C = \left\{ S \in A(F^*) : S = \{(T^*_{ij} \cup T^*_{kj}) \cup (T^*_{jk} \cup T^*_{ik}) \cup (T^*_{ki} \cup T^*_{ji})\} \right\}$$

Let $B \in A(F^*)$ such that $\left(v^*(B) - \sum_{j \in B} x^*(j)\right) = \tilde{e}(B,x^*)$ is maximal in C under the inequality $\geq$.

Claim (1): $(j \in B) \Rightarrow (i \in B)$

If not, then $\tilde{e}(B,x^*) = \tilde{S}^*_{ji}(x^*)$. By hypothesis, however, $\tilde{S}^*_{ij}(x^*) \gg_{\mu} \tilde{S}^*_{ji}(x^*)$ and for some $D \in C$, $\tilde{e}(D,x^*) \gg_{\mu} \tilde{e}(B,x^*)$ which is false by definition of B.

Claim (2): $(k \in B) \Rightarrow (j \in B)$

Same reasoning as claim (1).

Then $k \notin B$ else for some $S \in C$, where $i,j,k \in S$ which cannot be by construction of C. Also, $i \in B$ else $j \notin B$ and then $k \notin B$ which contradicts the construction of C as well.

Then $\tilde{S}^*_{ik}(x^*) = \tilde{e}(B,x^*) \gg_{\mu} \tilde{S}^*_{ki}(x^*)$ and thus one has $\tilde{S}^*_{ik}(x^*) \gg_{\mu} \tilde{S}^*_{ki}(x^*)$ while $x^*(k) >_{\mu} 0$ which is by definition $i\,\tilde{R}(x^*)\,k$.

Q.E.D.

In addition to the above properties we will make the assumption that the relation $\tilde{R}(x^*)$ is quasi-connected on the set of players. By quasi-connected we mean: $(\forall x^* \in (x^*,F^*))$ $(\forall i \in F^*)(\exists j \in F^*)(i\,\tilde{R}(x^*)\,j \;.v.\; j\,\tilde{R}(x^*)\,i)$. This assumption has the interpretation that there are no completely irrelevant players to the game, a player being irrelevant if no one is stronger than himself and he is stronger than no one. We require the assumption to avoid degenerate instances of stability in what follows. This assumption could conceivably be weakened to say that the set of irrelevant players is negligible.

Definition I.4: Corresponding to the relation $\tilde{R}(x^*)$, one has the following set:

$$\tilde{M}_i = \{x^* \in (x^*,F^*) : (\sim\exists j \in F^*)(i\,\tilde{R}(x^*)\,j)\}$$

The following lemmas are adapted from Peleg [3].

Lemma I.5: For any $i \in F^*$, then the following is true.

$\tilde{M}_i = \{x^* \in (x^*,F^*) : (\sim\exists j \in F^*)(i\,\tilde{R}^*(x^*)\,j)\} \neq \phi$ and is Q-closed.

Proof: If $\tilde{M}_i = \phi$, then $(\forall x^* \in (x^*,F^*)(\exists j \in F^*)(i\,R(x^*)\,j)$.

However, the set $\left\{x^* \in (x^*,F^*) : \{x^*(j) = 0\} \subseteq x^*\right\}$ provides a contradiction by property (3) of Lemma I.3.

Let $x^* \in \tilde{M}_i$, then an equivalent form of $\tilde{M}_i$ is $\tilde{M}_i = \bigcap_{j \in F^*} \{y^* : \sim(i\,\tilde{R}(x^*)\,j)\}$. Since F^* is internal, the intersection over F^*, of Q-closed sets is Q-closed. Therefore $\tilde{M}_i$ is closed.

Q.E.D.

Lemma I.6: If $x^* \in (x^*,F^*)$, then $\exists i \in F^*$ such that $x^* \in \tilde{M}_i$ and $x^*(i) \underset{\not\approx}{>} 0$.

Proof: If the first assertion is false, then $x^* \in \left[(x^*,F^*) - \left(\bigcup_{i \in F^*} \tilde{M}_i\right)\right]$ must be true. This implies that $(\forall i \in F^*)(\exists j \in F^*)(i\,R(x^*)\,j)$. By Lemma I.3 $\tilde{R}(x^*)$ is acyclic, and since F^* is internally *Finite, one has that $\left[(\exists i \in F^*)(\forall j \in F^*)\left(\sim(i\,R(x^*)\,j)\right)\right]$. This is contradictory.

If the second assertion were false, and $x^*(i) = 0$ while $x^* \in \tilde{M}_i$, then $(\sim\exists j \in F^*)(j\, \tilde{R}(x^*)\, i)$. But if $x^* \in \tilde{M}_i$, then $(\sim\exists j \in F^*)(i\, \tilde{R}(x^*)\, j)$. Then by conjunction, one has $(\forall j \in F^*)$ $\left[(\sim(i\, \tilde{R}(x^*)\, j)) \ldots (\sim(j\, \tilde{R}(x^*)\, i))\right]$. In this case, however, $\tilde{R}(x)$ is not Quasi-Connected.

Q.E.D.

Definition I.7: For each player $i \in F^*$, let the following function be defined: $\tilde{c}_i : (x^*, F^*) \times \tilde{M}_i \to R_+^*$ where $\tilde{c}_i(x^*) = d(x^*, \tilde{M}_i) = \inf_{y^* \in \tilde{M}_i} |x^* - y^*|$.

Lemma I.8: $\tilde{c}_i(x^*) \geq 0$ and is Q-continuous.

Proof: Self-evident.

Q.E.D.

Definition I.9: A point $\tilde{x}^* \in (x^*, F^*)$ is said to be Quasi-Balanced if $\tilde{c}_i(x^*) = 0$ a.e. in F^*. We mean by a.e. in F^* that the set $S \subseteq F^*$ for which $\tilde{c}_i(x^*) \neq 0$ for $i \in S$ is such that $\left[\frac{||S||}{||F^*||} = 0\right]_{\text{Mod } M_1}$.

Alternatively put, a point $\tilde{x}^* \in (x^*, F^*)$ is seen to be Quasi-Balanced if $\tilde{x}^* \in \bigcap_{j \in F^*-S} \tilde{M}_j$ where S is such that $\left[\frac{||F^*-S||}{||F^*||} = 1\right]_{\text{Mod } M_1}$. Then, by definition I.2, for any pair of players in $(F^* - \bar{S})$, with respect to $\tilde{x}^* \in \bigcap_{j \in F^*-S} \tilde{M}_j$, it must be the case that neither $i\, \tilde{R}(\tilde{x}^*)\, j$, nor $j\, \tilde{R}(\tilde{x}^*)\, i$; in

brief, any two players in $(F^* - S)$ are equipollent with respect to $\tilde{x}^*$, if $\tilde{x}^*$ is Quasi-Balanced. The set of Quasi-Balanced payoff configurations in (x^*, F^*) for a *Finite cooperative game Γ^*, we term the Quasi-Kernel, and use the symbol $QK^*(\Gamma^*)$ to indicate the set, $\{\tilde{x}^* \varepsilon (x^*, F^*) : i \circledequal j$ a.e. in $F^*\}$.

An extremely useful alternative characterization of $QK^*(\Gamma^*)$, which we shall employ in subsequent sections, can be obtained from the following theorem, the reference for which we thank Dr. Lloyd Shapley of RAND.

▶Theorem I.10: A payoff configuration $\tilde{x}^* \varepsilon (x^*, F^*)$ belongs to $QK^*(\Gamma^*)$ if and only if $\left(\tilde{S}^*_{ij}(\tilde{x}^*) = \tilde{S}^*_{ji}(\tilde{x}^*)\right)_{\text{Mod } M_1}$ a.e. in F^*.

Proof: We follow Maschler, Peleg, and Shapley [6] Theorem 2.7.

It is a relatively simple matter to show that the superadditivity of $v^*(\cdot)$ on $A(F^*)$ implies that Γ^* is S-monotonic, which is to say, that $v^*(\cdot)$ on $A(F^*)$ is such that $v^*(S) \leq v^*(T)$ whenever $S \subset T$, for $S, T \varepsilon A(F^*)$.

Now if $\tilde{x}^* \varepsilon QK(\Gamma^*)$, then $(\forall i, j \varepsilon B)(i \circledequal j)$ where $B = (F^* - S)$ for some $S \varepsilon A(F^*)$ such that $\left(\frac{\|S\|}{\|F^*\|} = 0\right)_{\text{Mod } M_1}$. Let those coalitions in B, not equal to B or ϕ, such that $\bar{e}(S, \tilde{x}^*)$ is maximal under $\geq$, be denoted as $D(\tilde{x}^*)$. Then,

$$D(\tilde{x}^*) = \left\{S \varepsilon (2^B - \{B, \phi\}) : \bar{e}(S, \tilde{x}^*) \geq \bar{e}(T, \tilde{x}^*) \text{ for } T \varepsilon (2^B - \{B, \phi\})\right\}.$$

Let $E = \bigcap\{S : S \in D(\tilde{x}^*)$, then by definition of $D(\tilde{x}^*)$, $B \notin E$. If $E \neq \phi$, then allow $i \in E$ and $\bar{j} \in B - E$. Then clearly, $\tilde{S}^*_{i\bar{j}}(\tilde{x}^*) \underset{\tau}{>>} \tilde{S}^*_{\bar{j}i}(\tilde{x}^*)$, and thus by Definition I.2, $\tilde{x}(\bar{j}) = 0$, since $\tilde{x}^* \in QK^*(\Gamma^*)$. Let U be arbitrary and in $D(\tilde{x}^*)$. Then,

$$\tilde{e}(U,\tilde{x}^*) = v^*(U) - \sum_{j \in U} \tilde{x}^*(j) = v^*(U) - \sum_{j \in B} \tilde{x}^*(j)$$

$$v^*(U) - \sum_{j \in B} \tilde{x}^*(j) \leq v^*(B) - \sum_{j \in B} \tilde{x}^*(j)$$

and by S-monotonicity,

$$\tilde{e}(U,\tilde{x}^*) \leq v^*(B) - \sum_{j \in B} \tilde{x}^*(j) \leq v^*(F^*) - \sum_{j \in F^*} \tilde{x}^*(j) = 0$$

Then if $\bar{j} \in B - U$, then $\tilde{e}(\{\bar{j}\},\tilde{x}^*) = 0$, and therefore $\{\bar{j}\} \in D(\tilde{x}^*)$. Since then both $\{\bar{j}\}$ and U are in $D(\tilde{x}^*)$, it must be that $E = \phi$. Then the assumption that $E \neq \phi$ is false and it must be the case that $E = \phi$, in fact.

Suppose next that for some $i,\bar{j} \in B$ $\left(\tilde{S}^*_{i\bar{j}}(\tilde{x}^*) \neq \tilde{S}^*_{\bar{j}i}(\tilde{x}^*)\right)_{\text{Mod } M_1}$. Without loss of generality let this imply that $\tilde{S}^*_{i\bar{j}}(\tilde{x}^*) \underset{\tau}{>>} \tilde{S}^*_{\bar{j}i}(\tilde{x}^*)$ and that therefore since $\tilde{x} \in QK^*(\Gamma^*)$ $\tilde{x}^*(\bar{j}) = 0$. Then for some coalition in $D(\tilde{x}^*)$, say U^0, $i \notin U^0$, by virtue of the fact that $E = \phi$. Then, by S-monotonicity, one obtains

$$e(U^0 \cup \{\bar{j}\},x^*) = v^*(U^0 \cup \{\bar{j}\}) - \sum_{j \in U^0} \tilde{x}^*(j) - \tilde{x}^*(\bar{j})$$

$$v^*(U^0 \cup \{\bar{j}\}) - \sum_{j \in U^0} \tilde{x}^*(j) - \tilde{x}^*(\bar{j}) \geq v^*(U^0) - \sum_{j \in U^0} \tilde{x}^*(j)$$

$$= \tilde{e}(U^0,\tilde{x}^*)$$

By definition of $\tilde{S}^*_{\bar{j}i}(\tilde{x}^*)$ and because $U^0 \in D(\tilde{x}^*)$, it cannot be the case that $\tilde{S}^*_{\bar{j}i}(\tilde{x}^*) \underset{\tau}{<<} \tilde{S}^*_{i\bar{j}}(\tilde{x}^*)$ as supposed. Then $\left(\tilde{S}^*_{ij}(x^*) = \tilde{S}^*_{ji}(\tilde{x}^*)\right)_{\text{Mod } M_1}$ for all $i,j \in B$, which is to say, a.e. in F^*,

since by completely symmetric reasoning, one can show that $\sim\left(\tilde{S}^{*}_{ji}(\tilde{x}^{*}) \gg_{i} \tilde{S}^{*}_{i\bar{j}}(\tilde{x}^{*})\right)$ for a given $\bar{j}, i \in B$.

The other direction of the theorem is immediate since by definition, if $\left(\tilde{S}^{*}_{ij}(\tilde{x}^{*}) = \tilde{S}^{*}_{ji}(\tilde{x}^{*})\right)_{\text{Mod } M_1}$ a.e. in F^*, then, i⊜j a.e. in F^*.

Q.E.D.

Theorem I.11: For a Q-bounded *Finite cooperative game $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$, there exists an $\tilde{x}^* \in (x^*, F^*)$ such that $\tilde{x}^* \in QK^*(\Gamma^*)$.

Proof: Let the following mapping be constructed, which is Q-continuous by Lemma I.8: $\tilde{f} : (x^*, F^*) \to (x^*, F^*)$ as

$$\tilde{f}(x^*)(j) = \frac{x^*(j) + \tilde{c}_j(x^*)}{1 + \sum_{j \in F^*} \tilde{c}_j(x^*)}$$

We will employ a version of a fixed point argument, to which the following lemmata are preliminary.

Lemma I.11.1: The set of payoff configurations (x^*, F^*) is Q-convex.

Proof: We can proceed by internal *Finite induction. For the case where $n = 2$, consider x^*, $y^* \in (x^*, F^*)$ and $\alpha \in (0,1)$. We wish to show that

$$\left(\alpha\left(\sum_{j \in F^*} x^*(j)\right) + (1-\alpha)\left(\sum_{j \in F^*} y^*(j)\right) = v^*(F^*)\right)_{\text{Mod } M_1}$$

By definition, $\sum_{j \varepsilon F^*} x^*(j) = v^*(F^*) + e_1$ and $\sum_{j \varepsilon F^*} y^*(j) = v^*(F^*) + e_2$ for $e_1, e_2 \varepsilon M_1$. Then, one obtains

$$\alpha\left(\sum_{j\varepsilon F^*} x^*(j)\right) + (1-\alpha)\left(\sum_{j\varepsilon F^*} y^*(j)\right) = \alpha\left(v^*(F^*)+e_1\right)+(1-\alpha)\left(V^*(F^*)+e_2\right).$$

However, $\alpha\left(v^*(F^*)+e_1\right)+(1-\alpha)\left(v^*(F^*)+e_2\right) = v^*(F^*) + \alpha e_1 + (1-\alpha)e_2$. But for $e_1, e_2 \varepsilon M_1$, $(\alpha e_1 + (1-\alpha)e_2) \varepsilon M_1$ for $\alpha \varepsilon (0,1)$. Then $\alpha\left(v^*(F^*)+e_1\right) + (1-\alpha)\left(v^*(F^*)+e_2\right) = v^*(F^*) + e_3$ for $e_3 \varepsilon M_1$, $e_3 = \alpha\varepsilon_1 + (1-\alpha)e_2$ and therefore one obtains

$$\left[\alpha\left(v^*(F^*)+e_1\right) + (1-\alpha)\left(v^*(F^*)+e_2\right) = v^*(F^*)\right]_{\text{Mod } M_1}.$$

But the last expression is simply that

$$\left[\alpha\left(\sum_{j\varepsilon F^*} x^*(j)\right) + (1-\alpha)\left(\sum_{j\varepsilon F^*} y^*(j)\right) = v^*(F^*)\right]_{\text{Mod } M_1}.$$

Assume the conclusion to be true for the case $n-1$ for $n \varepsilon N^*$. Then choose $\alpha_j \varepsilon (0,1)$ for $j = 1, \ldots, n$ such that $\sum_{j=1}^{n} \alpha_j = 1$. Then by the hypothesis of the induction step, $z^* = \left(\sum_{j=1}^{n-1} \alpha_j x_j^*\right) \Big/ (1-\alpha_n)$ for $x_j^* \varepsilon (x^*,F^*)$ is such that $z^* \varepsilon (x^*,F^*)$, since $\sum_{j=1}^{n-1}\alpha_j \Big/ (1-\alpha_n) = 1$. But then, for $x_n^* \varepsilon (x^*,F^*)$, one observes that $\sum_{j=1}^{n} \alpha_j x_j^* = \alpha_n x_n^* + (1-\alpha_n) z^*$. Then by the basis of the induction

$$\left(\sum_{j=1}^{n} \alpha_j x_j^*\right) \varepsilon (x^*,F^*).$$

Q.E.D.

Lemma I.11.2: The set of payoff configurations is Q-closed.

Proof: It will suffice to show that (x^*,F^*) is closed under F-limits. If $\Phi : N^* \to (R^*)^{F^*}$ is a sequence on N^* with values in $(R^*)^{F^*}$ then z^* is an F-limit of Φ, or Φ F-converges to z^*

if it is the case that

$(\forall\delta \in R_+ - \{0\})(\exists n \in N)(\forall m \in N)(m \geq n \Rightarrow |\phi(m) - z^*| < \delta)$

Then suppose $\{x_j^*\}_{j \in N^*}$ is a sequence such that each $x_j^* \in (x^*,F^*)$, and let z^* be an F-limit of $\{x_j^*\}_{j \in N^*}$, but $z^* \notin (x^*,F^*)$. Then $\left(\sum_{j\in F^*} z^*(j)\right) \notin u\left(v^*(F^*)\right)$ which means that $\left|\sum_{j\in F^*} z^*(j) - v^*(F^*)\right| > \delta$ for some $\delta \in R_+ - \{0\}$. But if z^* is an F-limit of $\{x_j^*\}_{j \in N^*}$, for m sufficiently large, $x_m^* \in u(z^*)$.
But in that case we obtain

$$\left|\sum_{j\in F^*} z^*(j)-v^*(F^*)\right| \leq \left|\sum_{j\in F^*} z^*(j)- \sum_{j\in F^*} x_m^*(j)\right|+\left|\sum_{j\in F^*} x_m^*(j)-v^*(F^*)\right|$$

and, therefore, that

$$\left|\sum_{j\in F^*} z^*(j)-v^*(F^*)\right| \leq e_1 + e_2 \qquad \text{for } e_1,\ e_2 \in M_1^+.$$

This contradicts the assumption that $z^* \notin (x^*,F^*)$.

Q.E.D.

Then in the light of the above two lemmas, since (x^*,F^*) is Q-bounded, normalization by $v^*(F^*)$ permits us to regard the set of payoff configurations as a Q-closed simplex of internal *Finite dimension in $(R_+^*)^{F^*}$. Is is then possible to employ the following result.

▶**Lemma I.11.3:** Let E be a *Finite Q-closed simplex of internal dimension in $(R_+^*)^{F^*}$. Let $\bar{f} : E \to E$ be a Q-continuous mapping. For $\hat{a} \in E$, let $A_{\hat{a}} = \left\{c(f(B(\hat{a}))) :\right.$ $\left.B = \{y \in E : |\hat{a} - y| < r \text{ for } r \in R_+^* - \{0\}\}\right\}$ where $c(\cdot)$ denotes the Q-convex closure. Then for some $\hat{a} \in E$, $\hat{a} \in A_{\hat{a}}$.

Proof: Machover and Hirschfeld, Lectures in Nonstandard Analysis, Springer Verlag, 1969.

Then Lemmas I.11.1, I.11.2, and I.11.3 with $\tilde{f}$ defined as before, upon interpretation, allowing B to be defined as $u(\hat{a})$, where $u(\hat{a})$ is the monad of $\hat{a}$ defined as $u(\hat{a}) = \bigcap_{\hat{a}\varepsilon U_v^*} U_v^*$, and U_v^* is an open S-ball containing $\hat{a}$, one obtains $\tilde{x}^* \varepsilon (x^*, F^*)$, for which it is true that $\tilde{x}^* \varepsilon A_{\tilde{x}^*} = \left\{c\left[\tilde{f}\left((u(\tilde{x}^*)\right)\right]\right\}$. Then $\tilde{x}^* = \tilde{f}(y^*)$ for $\tilde{f}(y^*) \varepsilon c\left[\tilde{f}\left(u(\tilde{x}^*)\right)\right]$. The latter expression means that $\left[\tilde{f}(y^*) = \tilde{f}(\tilde{x}^*)\right]_{\text{Mod } M_1}$, from whence $\left[\tilde{x}^* = \tilde{f}(\tilde{x}^*)\right]_{\text{Mod } M_1}$. Then since

$$\tilde{f}(\tilde{x}^*)(j) = \frac{\tilde{x}^*(j) + \tilde{c}_j(\tilde{x}^*)}{1 + \sum\limits_{j\varepsilon F^*} \tilde{c}_j(\tilde{x}^*)}$$

and $\tilde{x}^*(j) \varepsilon M_0^+$ for any $j \varepsilon F^*$, it follows that

$$\left[\tilde{x}^*(j) = \frac{\tilde{x}^*(j) + \tilde{c}_j(\tilde{x}^*)}{1 + \sum\limits_{j\varepsilon F^*} \tilde{c}_j(\tilde{x}^*)}\right]_{\text{Mod } M_1}$$

which equivalently stated is,

$$\left[\tilde{x}^*(j)\left[1 + \sum_{j\varepsilon F^*} \tilde{c}_j(\tilde{x}^*)\right] = \tilde{x}^*(j) + \tilde{c}_j(\tilde{x}^*)\right]_{\text{Mod } M_1}$$

This last expression in turn implies

$$\left[\tilde{x}^*(j)\left[\sum_{j\varepsilon F^*} \tilde{c}_j(\tilde{x}^*)\right] = \tilde{c}_j(\tilde{x}^*)\right]_{\text{Mod } M_1}$$

which is true for $\tilde{c}_j(\tilde{x}^*) \varepsilon M_1$, for all $j \varepsilon F^*$ only if $\tilde{c}_j(\tilde{x}^*) \not\geq 0$ for $j \varepsilon S \subseteq F^*$ such that $\left[\frac{\|S\|}{\|F^*\|} = 0\right]_{\text{Mod } M_1}$, that is, only if $\tilde{x}^*$ is Quasi-Balanced.

Q.E.D.

II. A *Finite Game for which $K^*(\Gamma^*) = \phi$ and $QK^*(\Gamma^*) \neq \phi$

Allow the game $\Gamma^*_N(F^*,v^*)$ to be defined as follows for $F^* = [0,\omega]$, $\omega \in N^* - N$

$$v^*(S) = \begin{cases} 1 & \text{for } S \in A(F^*) \text{ and } N(K) \subseteq S \\ 0 & \text{else} \end{cases}$$

By $N(K)$ is meant the set $\{n \in N : n \geq K\}$ for N the standard natural numbers and $K \in N$. Denote by $K^*(\Gamma^*)$, the syntactic equivalent of the Kernel for standard finite games in M^*. The following theorem serves to indicate that $QK^*(\Gamma^*)$ is a strictly weaker concept than $K^*(\Gamma^*)$ for *Finite cooperative games.

Theorem II.1: For the game $\Gamma^*_N(F^*,v^*)$, $K^*(\Gamma^*) = \phi$.

Proof: Assume that $\tilde{x}^* \in K^*(\Gamma^*)$. Then it must be the case that $\sum_{j \in F^*} \tilde{x}^*(j) = 1$, since $F^* \supseteq N$. Therefore $\tilde{x}^* = (\tilde{x}^*(1), \ldots, \tilde{x}^*(\omega))$ cannot be identically zero. While it may be true that $\tilde{x}^*(j) \gg 0$ for no $j \in F^*$, one can establish the following:

Lemma II.1.2: There is a least standard integer $\tilde{K} \in N$, for which $\tilde{x}^*(\tilde{K}) \underset{\ne}{>} 0$.

Proof: If not, then for some $i \in F^* - N$, and all $j \in N$, $\tilde{x}^*(i) > \tilde{x}^*(j)$, else $\tilde{x}^* = 0$. We show this leads to a contradiction.

Definition II.1.2.1: For a game $\Gamma^*(F^*,v^*)$, $K \, D \, \ell$, read K is more desirable than ℓ, if

$$v^*(S \cup \{K\}) \geq v^*(S \cup \{\ell\}) \text{ whenever } \ell, K \notin S \in A(F^*)$$

A payoff $\bar{x}^* \in (x^*,F^*)$ is said to preserve desirability if $K \, D \, \ell \Rightarrow \bar{x}^*(K) \geq \bar{x}^*(\ell)$.

▶**Lemma II.1.2.2: If $\bar{x}^* \in K^*(\Gamma^*)$, then $\bar{x}^*$ preserves desirability (Maschler and Peleg, 1966 [2]).**

Proof: If $\bar{x}^* \in K^*(\Gamma^*)$ and $K \, D \, \ell$, but $\bar{x}^*(K) < \bar{x}^*(\ell)$, choose $S \in A(F^*)$ so that $\bar{S}^*_{\ell,K}(\bar{x}^*) = \bar{e}(S,\bar{x}^*)$. Allow $T = (S \cup \{K\} - \{\ell\})$. Then $v^*(T) \geq v^*(S)$ since $k \, D \, \ell$. Then because $\bar{x}^*(k) < \bar{x}^*(\ell)$, one has $\bar{e}(T,\bar{x}^*) > \bar{e}(S,\bar{x}^*)$. Then $\bar{S}^*_{K\ell}(\bar{x}^*) > \bar{S}^*_{\ell K}(\bar{x}^*)$ and $\bar{x}^*(\ell) > \bar{x}^*(K) \geq 0$. But then $K > \ell$ and therefore $\bar{x}^* \notin K^*(\Gamma^*)$.

Q.E.D.

To establish Lemma II.1.2, note that if for some $i \in F^* - N$ and all $j \in N$, $\bar{x}^*(i) > \bar{x}^*(j)$, by the contrapositive of Lemma II.1.2.2, for some $S \in A(F^*)$ it must be that $v^*(S \cup \{i\}) > v^*(S \cup \{j\})$, where $i,j \notin S$. However, for Γ^*_N, any S not containing i,j has value 0 or 1, and that value is unchanged with or without either i or j for $i \in F^* - N$ and $j \in N$. Therefore there can be no $S \in A(F^*)$ such that $v^*(S \cup \{i\}) > v^*(S \cup \{j\})$ for $i,j \notin S$ and $i \in F^* - N$ and $j \in N$.

Q.E.D.

Lemma II.1.3: $v^*(S)$ is superadditive.

Proof: If $v^*(S)$ is not superadditive, then for some internal $S \subseteq F^*$, $S = S_1 \cup S_2$, $S_1 \cap S_2 = \phi$, it must be the case that $v^*(S) < v^*(S_1) + v^*(S_2)$.

It is either the case that $N(K) \subseteq S$ for some $K \in N$, or $N(K) \not\subseteq S$ for any $K \in N$.

If $N(K) \not\subseteq S$ for any $K \in N$, then $v^*(S) = 0$. For $v^*(S) < v^*(S_1) + v^*(S_2)$ to obtain, it must be the case that $[v^*(S_1) \vee v^*(S_2)] = 1$. Then either $S_1 \supseteq N(K_1)$ or $S_2 \supseteq N(K_2)$ for K_1, $K_2 \in N$. Then $S \supseteq N(K)$, $K = K_1$ or K_2.

If $S \supseteq N(K)$ for some $K \in N$, then $v^*(S) = 1$. For $v^*(S) < v^*(S_1) + v^*(S_2)$ to obtain, then it must be the case that $[v^*(S_1) \wedge v^*(S_2)] = 1$. Then $S_1 \supseteq N(K_1)$ and $S_2 \supseteq N(K_2)$ for K_1, $K_2 \in N$. But then, $N(K) \subseteq (S_1 \cap S_2)$ for $\max\{K_1, K_2\} = K$.

Q.E.D.

If $\tilde{x}^* \in K^*(\Gamma^*)$, one requires, by virtue of the superadditivity of $v^*(S)$, $\tilde{S}^*_{\tilde{K},\tilde{K}+1}(\tilde{x}^*) = \tilde{S}^*_{\tilde{K}+1,\tilde{K}}(\tilde{x}^*)$ by reasoning analogous to p. 17 footnote (**) and Definition 3.1 with Theorem 3.3 of [2].

Obviously, $[\sup_{S \in T^*_{\tilde{K},\tilde{K}+1}} v^*(S)] = 0$, while $[\sup_{S \in T^*_{\tilde{K}+1,\tilde{K}}} v^*(S)] = 1$ where $T^*_{\tilde{K}+1,\tilde{K}}$ is the collection of sets in $A(F^*)$ containing $\tilde{K}+1$ but not $\tilde{K}$. Then the requirement that $\tilde{S}^*_{\tilde{K}+1,\tilde{K}}(\tilde{x}^*) = \tilde{S}^*_{\tilde{K},\tilde{K}+1}(\tilde{x}^*)$ entails $-\tilde{x}^*(\tilde{K}) = 1 - \sum_{j=\tilde{K}+1}^{\omega} \tilde{x}^*(j)$. Equivalently, we have $-2\tilde{x}^*(\tilde{K}) = 0$. But this cannot be by choice of $\tilde{K}$ in Lemma II.1.2. This establishes the theorem.

Q.E.D.

On the other hand, for $\tilde{x}^* \varepsilon\, QK^*(\Gamma^*)$, it is required as a necessary condition that given $\tilde{K}$ as above,

$\left(\tilde{S}^*_{\tilde{K},\tilde{K}+1}(\tilde{x}^*) = \tilde{S}^*_{\tilde{K}+1,\tilde{K}}(\tilde{x}^*)\right)_{\text{Mod } M_1}$ a.e. in F^*, in which case,

$\left(-\tilde{x}^*(\tilde{K}) = 1 - \sum_{j=\tilde{K}+1}^{\omega} \tilde{x}^*(j)\right)_{\text{Mod } M_1}$ and therefore that

$\left(-2\tilde{x}^*(\tilde{K}) = 0\right)_{\text{Mod } M_1}$ which for $\tilde{x}^*(\tilde{K}) \underset{\not\approx}{>} 0$ does not entail a contradiction for $\tilde{x}^*(\tilde{K}) \underset{\not\approx}{>} 0$, a nonzero positive infinitesimal. It can be easily verified that any symmetric vector $\tilde{x}^* = (1/n, \ldots, 1/n)$ for $n \varepsilon N^* - N$ such that $\left(\frac{\|n\|}{\|\omega\|} = 1\right)_{\text{Mod } M_1}$ is in $QK^*(\Gamma^*)$ for the game Γ^*_N defined as above.

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13. ABSTRACT

A solution concept analogous to the Kernel of a finite cooperative N-person game of the classical Von Neumann-Morgenstern variety is defined for a class of nonstandard games. The solution concept treated, the Quasi-Kernel, is shown to be in general nonempty and represents a strictly weaker solution concept than the syntactic equivalent of the Kernel as defined for standard cooperative games in the *Finite context.

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